9.9 Applications and Deriv. Graphs

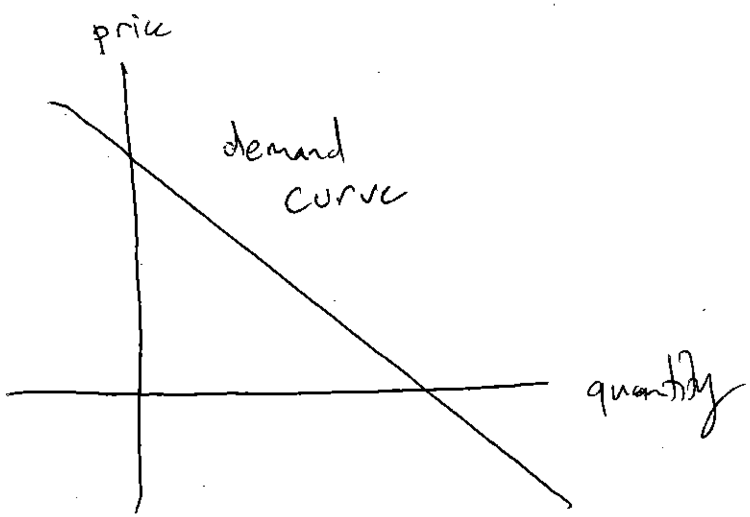
Application 1: Demand Functions and Total Revenue

Recall from Math 111:

A *demand curve is* given by an equation that relates the *quantity* that will sell based on the market *selling price*.

And

$$\text{Revenue} = \text{Price} \cdot \text{Quantity}$$



Example (from HW 9.9/1):

Assume x = items sold (quantity)

p = price per item.

From market analysis you estimate

$$p = 460 - 0.2x$$

a. Find TR(x) and MR(x).

b. What quantity maximizes TR(x)?

(a) $TR(x) = p \cdot x = (460 - 0.2x)x$

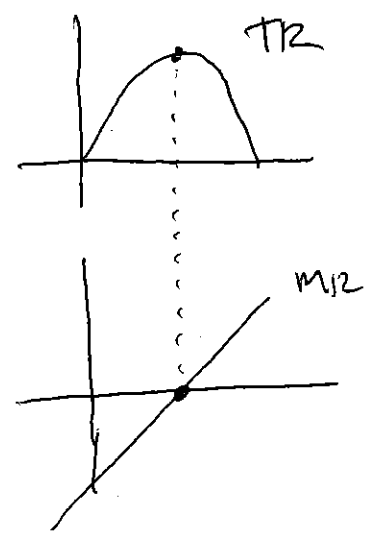
$$\boxed{TR(x) = 460x - 0.2x^2}$$

$$\boxed{MR(x) = 460 - 0.4x} \overset{?}{=} 0$$

(b) $460 - 0.4x \overset{?}{=} 0$

$$x = \frac{460}{0.4}$$

$$\boxed{x = 1150 \text{ items}}$$



Two More Examples:

1. (part of HW 9.9/6)
   The selling **price** on the competitive market is 90 dollars/item.
   Find $TR(x)$ and $MR(x)$.

$$TR(x) = px = 90x$$

$$MR(x) = TR'(x) = 90 \ \frac{\text{dollars}}{\text{item}}$$

EACH ADDITIONAL ITEM BRINGS IN \$90.

2. If the demand function is

$$p = \frac{500}{(3x+1)^2}$$

Find $TR(x)$ and $MR(x)$.

$$TR(x) = px = \frac{500}{(3x+1)^2} \cdot x$$

$$\Rightarrow TR(x) = \frac{500x}{(3x+1)^2}$$

$$MR(x) = \frac{(3x+1)^2 \cdot 500 - 500x \cdot 2(3x+1)(3x+1)'}{(3x+1)^4}$$

$$= \frac{500(3x+1)\left[(3x+1) - 6x\right]}{(3x+1)^4}$$

$$= \frac{500(3x+1)(-3x+1)}{(3x+1)^4}$$

COULD CANCEL ONE!

$$= \boxed{\frac{500(-3x+1)}{(3x+1)^3}}$$

***Application 2*: Cost Analysis and Profit**

Recall from Math 111:

TC(x) = "total cost to produce x items"

AC(x) = "overall average cost to produce *x* items"

$$AC(x) = \frac{TC(x)}{x} \quad \text{and } TC(x) = x\, AC(x)$$

Example (part of HW 9.9/5 and 6):

The (average) **cost** per unit is given by

$$130 + 0.5x \quad \text{dollars/item}$$

Find $TC(x)$ and $MC(x)$.

$$TC(x) = x \cdot AC(x)$$

$$= x \cdot (130 - 0.5x)$$

$$\boxed{TC(x) = 130x + 0.5x^2}$$

$$\boxed{MC(x) = 130 + x}$$

Recall from Math 111:

Profit and marginal profit are given by

$$P(x) = TR(x) - TC(x)$$

$$MP(x) = MR(x) - MC(x)$$

When profit is maximized

$$MR(x) = MC(x)$$

Specifically, where it switches from MR > MC to MR < MC.

Example (*directly* from HW 9.9/5)

The price of a certain product is $400. The cost per unit of producing the product is 130 + 0.5x dollars/item.

a. Find $TR(x)$ and $MR(x)$.
b. Find $TC(x)$ and $MC(x)$.
c. Find $P(x)$ and $MP(x)$.
d. How many units should you produce and sell to maximize its profits?

$TR(x) = 400x$, $MR(x) = 400$

$TC(x) = 130x + 0.5x^2$, $MC(x) = 130 + x$

$P(x) = (400x) - (130x + 0.5x^2)$

$= 400x - 130x - 0.5x^2$

$P(x) = 270x - 0.5x^2$

$MP(x) = 270 - x$

d) When $MR(x) = MC(x)$

$400 = 130 + x$

$\Rightarrow x = \boxed{270 \text{ items}}$ occurs here

$\Rightarrow P(270)$

$= 270(270) - 0.5(270)^2$

$= \boxed{\$36{,}450}$ max profit

Another example
(*directly* from an old midterm):
You sell items.
If q is in **<u>hundred</u> items**, then TR(q) and TC(q) in **<u>hundred</u> dollars** are given by

$$TR(q) = 30q$$
$$TC(q) = q^3 - 15q^2 + 78q + 10$$

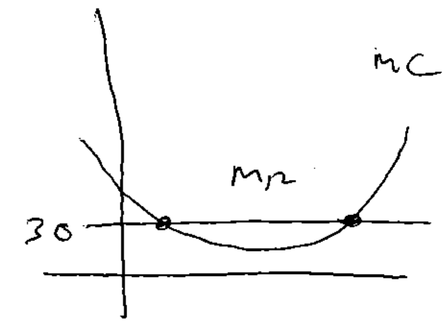
a. Find marginal cost at 2 hundred items

$$MC(q) = TC'(q)$$
$$= 3q^2 - 30q + 78$$
$$MC(2) = 3(2)^2 - 30(2) + 78$$
$$= 30 \quad \$/\text{item}$$

("The 201st item costs about \$30 to produce")

b. Find the longest interval over which marginal revenue exceeds marginal cost.

$\boxed{\text{STEP 1}}$ FIND WHEN THEY ARE EQUAL!



$$MR(q) = MC(q)$$
$$\Updownarrow$$
$$30 = 3q^2 - 30q + 78 \quad \div 3$$
$$\Rightarrow 0 = 3q^2 - 30q + 48 \Rightarrow 0 = q^2 - 10q + 16$$
$$q = \frac{10 \pm \sqrt{10^2 - 4(1)(16)}}{2}$$
$$= \frac{10 \pm \sqrt{100 - 64}}{2} = \frac{10 \pm 6}{2}$$
$$= 4/2 = 2 \text{ or } 16/2 = 8$$

$\boxed{\text{STEP 2}}$ INTERPRET

$$\boxed{q = 2 \text{ to } q = 8}$$

MAX PROFIT OCCURS HERE!

c. What is the maximum value of profit?

$$P(q) = TR(q) - TC(q)$$

$$= 30q - (q^3 - 15q^2 + 78q + 10)$$

$$= 30q - q^3 + 15q^2 - 78q - 10$$

$$P(q) = -q^3 + 15q^2 - 48q - 10$$

$$P(8) = -(8)^3 + 15(8)^2 - 48(8) - 10$$

$$\boxed{54 \text{ hundred dollars}}$$

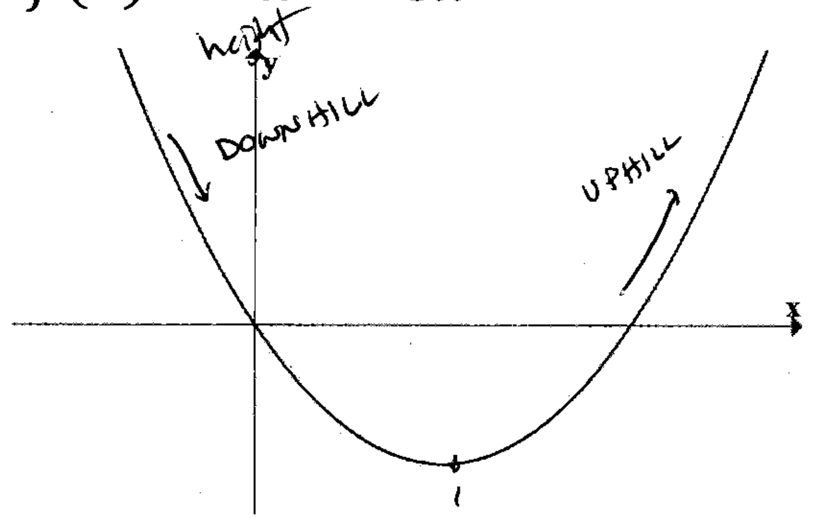
Graphs and Derivatives

Example: Let $f(x) = 2x^2 - 3x$

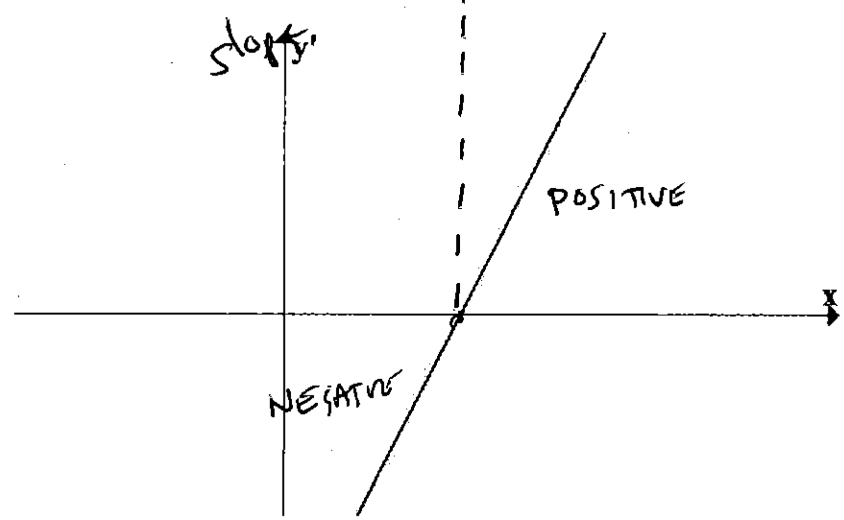
Find $f'(x)$.

$f'(x) = 4x - 3$

$$f(x) = 2x^2 - 3x$$



$$f'(x) = 4x - 3$$



Notes/Observations: Given $y = f(x)$.

- $y = f'(x)$ is a new function.
- $f(x)$ = "height of the graph at *x*"
- $f'(x)$ = "slope of $f(x)$ at x"
- $f'(x)$ is "instantaneous rate of change" (speedometer speed)
- The units of $f'(x)$ are $\frac{\text{y-units}}{\text{x-units}}$.

Fundamental to all applications:

$f(x)$	$f'(x)$
Horiz. Tangent *(peak, valley, or "chair")*	**Zero** *(crosses x-axis)*
Increasing (*uphill*)	**Positive** (*above x-axis*)
Decreasing (*downhill*)	**Negative** (*below x-axis*)

Old Exam Question:

The height of a balloon after t seconds is given by

$$B(t) = 15t^2 - t^3 \qquad \text{feet.}$$

a. At time t = 1 second, is the balloon rising or falling?
b. Find the maximum height reached by the balloon.

(a) $B'(t) = 30t - 3t^2$

$B'(1) = 30(1) - 3(1)^2 = 27$

POSITIVE!

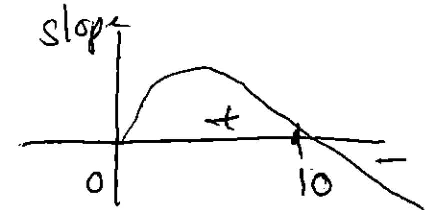
RISING

(b) $B'(t) = 30t - 3t^2 \overset{?}{=} 0$

$10t - t^2 = 0$

$t(10 - t) = 0$

$t = 0$ or $t = 10$



From $t=0$ to $t=10$, B RISING.
AFTER $t=10$, B FALLING.

MAX HEIGHT OCCURS AT $t=10$

$B(10) = 15(10)^2 - (10)^3$

$= \boxed{500 \text{ feet}}$